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Birth of a strange nonchaotic attractor: A renormalization group analysis

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A renormalization group analysis is developed for a quasiperiodically forced nonlinear system near the onset of a strange nonchaotic attractor. The scaling properties for such an attractor are found and verified numerically.

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Strange nonchaotic attractors typically appear in quasiperiodically forced nonlinear systems. This type of attractor was described by Grebogi *et al.* [1] and has been studied both theoretically and experimentally [2–9]. Strange nonchaotic attractors have only negative Lyapunov exponents (besides the zero one connected to the quasiperiodic forcing), but they are geometrically strange (fractal). One example of a strange nonchaotic attractor appears in the theory of the Schrödinger equation with a quasiperiodic potential. Bondeson *et al.* [10] showed that the transition to localization in the Schrödinger equation corresponds to the appearance of a strange nonchaotic attractor in the related dynamical system. Another example of such a behavior was obtained in a nonlinear mechanical system with quasiperiodic forcing [7]. Most studies of strange nonchaotic attractors have been devoted to their characterization as fractal objects. However, to the best of our knowledge, no quantitative analysis of the transition to a strange nonchaotic attractor has been made. To analyze this problem we develop in this paper a renormalization group (RG) approach. It is known that such an approach is a powerful tool in studies of transitions to chaos [11].

The first model, for which a strange nonchaotic attractor was reported [1], is a two-dimensional map

$$x_{i+1}=f(x_i,\theta_i)=2\lambda(\tanh x_i)\sin(2\pi\theta_i), \tag{1}$$

$$\theta_{i+1}=\theta_i+\omega \quad \mathrm{mod}1. \tag{2}$$

The circle map (2) defines (for irrational ω) a quasiperiodic force which is multiplicative in the nonlinear equation (1). Grebogi *et al.* [1] have shown that a strange nonchaotic attractor exists for $\lambda>1$, while for $\lambda<1$ the attractor is the line $x=0$, which is nonstrange. So, the transition occurs at $\lambda=1$.

To achieve clearer presentation, we consider a modification of the map and take instead of (1)

$$x_{i+1}=2\lambda\frac{x_i}{(1+x_i^2)^{1/2}}\sin(2\pi\theta_i). \tag{3}$$

The nonlinearity in Eq. (3) is similar to that in (1), so all the arguments of Ref. [1] can be easily reproduced for Eq. (3): for $\lambda<0$ the attractor is the line $x=0$, and for $\lambda>1$ a strange nonchaotic attractor appears. In this paper we study the model (2),(3) in a vicinity of the critical parameter value $\lambda_c=1$. We choose here the rotation number as the reciprocal of the golden mean: $\omega=1/W=(\sqrt{5}-1)/2$.

As the first step, we introduce instead of x two new variables C and z, such that $x=Cz$, where z obeys the recurrent relation

$$z_{i+1}=2z_i\sin(2\pi\theta_i), \quad z_0=1. \tag{4}$$

The equation for C is then

$$C_{i+1}=\lambda\frac{C_i}{(1+z_i^2C_i^2)^{1/2}}. \tag{5}$$

The system (4),(5) is completely equivalent to (3), but now we have separated the rapidly oscillating component z and the slowly varying "amplitude" C. [Note that Eq. (4) appears if we neglect the nonlinearity in (1) or (3) and fix the parameter λ at its critical value.] Equation (5) is nonlinear, depends on λ, and therefore describes the "bifurcation" of the birth of a strange nonchaotic attractor.

Let us now apply an RG approach to this problem. As usual in the context of nonlinear dynamics, this approach suggests the consideration of the evolution of the system during progressively increasing intervals of discrete time i [11–15]. Because we have chosen the rotation number to be the reciprocal of the golden mean, it seems worth taking these time intervals as the Fibonacci numbers $F_0=0$, $F_1=F_2=1$, $F_3=2$, ..., $F_n=F_{n-1}+F_{n-2}$ (see Refs. [12–15]. It is easy to check that the nonlinear function in Eq. (5) has the following property: if $F(x)=x(1+Ax^2)^{-1/2}$ and $G(x)=x(1+Bx^2)^{-1/2}$, then $F(G(x))=x[1+(A+B)x^2]^{-1/2}$. Using this, we can represent the evolution of our variables during F_n time steps for the critical parameter value $\lambda_c=1$ as

$$z_{F_n}=P_n(\theta)z_0, \tag{6}$$

$$C_{F_n}=C_0[1+S(\theta)C_0^2]^{-1/2}, \tag{7}$$

$$\theta_{F_n}=\theta+F_n\omega \quad \mathrm{mod}1. \tag{8}$$

Here we denote the initial phase at $i=0$ as θ (without index) and introduce two functions of this initial phase θ

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TABLE I. Periods of the RG equation solutions and scaling indices for some initial phases θ^0.

θ^0	Period of Q^2	α	γ	Slopes in Fig. 3
0	3	1.3239	0.1943	
1/4	6	7.4246	0.6943	0.695
$\omega=(\sqrt{5}-1)/2$	3	31.4512	1.1943	1.193
$(3\sqrt{5}-5)/10$	12	55.1243	0.6943	0.694
$(5\sqrt{5}-7)/19$	9	55.7639	0.9284	0.928

$$P_n(\theta)=\prod_{i=0}^{F_n-1} 2\sin[2\pi(i\omega+\theta)],$$

$$S_n(\theta)=\sum_{i=0}^{F_n-1} z_i^2=\sum_{i=0}^{F_n-1}\prod_{j=0}^{i-1} 4\sin^2[2\pi(j\omega+\theta)]. \tag{9}$$

It is possible to calculate the functions $P_n(\theta)$ and $S_n(\theta)$ recursively. Taking into account that $F_{n+2}=F_{n+1}+F_n$, we can obtain P_{n+2} and S_{n+2} by considering separately the contributions from the first F_{n+1} and the second F_n iterations [16]. Note that after F_{n+1} steps the new values of the variables θ and z (correspondingly F_{n+1} and P_{n+1}^2) appear as the initial values of θ_i and z_i for the second F_n iterations. Thus P_{n+2} and S_{n+2} are expressed as

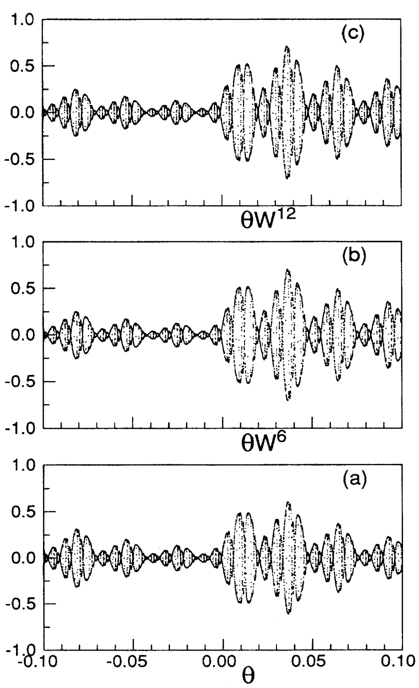

FIG. 1. The phase portraits of strange nonchaotic attractors in the system (3),(2) with $\theta^0=1/4$ for $\lambda=1+\varepsilon_0$ (a), $\lambda=1+\varepsilon_0\omega^6$ (b), and $\lambda=1+\varepsilon_0\omega^{12}$ (c), where $\varepsilon_0=0.01$. The values of θ are scaled near the point $\theta^0=1/4$ with factor ω^6; the values of x are scaled with the corresponding factor $\alpha=7.424$ (see Table I).

$$P_{n+2}(\theta)=P_{n+1}(\theta)P_n(\theta_{F_{n+1}}), \tag{10}$$

$$S_{n+2}(\theta)=S_{n+1}(\theta)+P_{n+1}^2(\theta)S_n(\theta_{F_{n+1}}). \tag{11}$$

Using a known relation for the Fibonacci numbers $F_{n+1}\omega=F_n-(-\omega)^{n+1}$ and the periodicity of all relevant functions of θ, we can represent $\theta_{F_{n+1}}$ in (10),(11) as

$$\theta_{F_{n+1}}=\theta+F_{n+1}\omega=\theta-(-\omega)^{n+1}. \tag{12}$$

Now let us try to renormalize the variable θ in order to get a reasonable asymptotic behavior of the functions P and S. From (12) it follows that the characteristic scale in θ decreases with n roughly as ω^n [this can be also seen from (9): the number of harmonics in the function P_n is equal to F_n, so the length scale can be estimated as $F_n^{-1}\approx\omega^n$]. We take some θ^0 as an origin, rewrite (10) and (11) with $\theta-\theta^0$ as the argument, and introduce a new phase variable $y=(\theta-\theta^0)/(-\omega)^n$. Then we define the scaled functions $Q_n(y)=P_n(y(-\omega)^n)$ and $H_n(y)=S_n(y(-\omega)^n)$. According to these definitions we obtain

$$Q_{n+2}(y)=Q_{n+1}(-\omega y)Q_n(\omega^2 y+\omega), \tag{13}$$

$$H_{n+2}(y)=H_{n+1}(-\omega y)+Q_{n+1}^2(-\omega y)H_n(\omega^2 y+\omega). \tag{14}$$

Note that Eq. (13) is independent of Eq. (14); moreover, the second one is linear. To study numerically the evolution of the functions Q and H we have approximated them by finite polynomial expansions and then performed an iterative computation of the coefficients according to Eqs. (13) and (14). Starting from the initial functions $Q_0=1$ and $Q_1(y)=-2\sin[2\pi\omega(y+\theta^0)]$ we have found that the dynamics of Q appears to depend subtly on θ^0. For some θ^0, the iterations result in a periodic behavior of Q_n, while for other initial phases these functional sequences behave in a random manner. We present some initial phases giving periodic behavior in Table I (a more detailed study will be presented elsewhere [17]). Note that for the cases $\theta^0=0$, ω, and $(\sqrt{5}-7)/19$, the period of Q is twice as that of Q^2, due to a symmetry of the solutions. Substituting the periodic solution of Eq. (13) into (14), we get a linear periodically forced equation. Hence it produces an exponential growth with some factor α^2 over a period (although the growth may generally be combined with modulation, we have observed only monotonous growth). The factors α found numerically are also given in Table I. It is important to note that the factor α depends on θ^0. The value α describes the growth of S_n. Therefore, according to Eq. (7) it is also the scaling factor for the amplitude C along the orbit starting at the chosen θ^0. The corresponding scaling factor for time is W^N, where N is the period of the RG cycle for Q^2 and $W=\omega^{-1}$. One evident conclusion from the analysis outlined above concerns the temporal behavior of the value of C exactly at the critical value λ_c: When time is increased by a factor W^N the value of C decreases by a factor α corresponding to the chosen initial phase. Thus the amplitude C decays as a power of time with an exponent $\gamma=\ln(\alpha)/(N\ln W)$ (cf. Table I) [18].

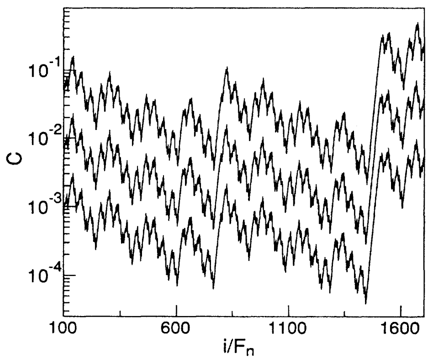

FIG. 2. Amplitude C vs time for the initial phase $\theta^0 = 1/4$ at the same λ as in Fig. 1 (decreasing from top to bottom). The time axis is scaled for these λ with factors E_6, F_{12}, and F_{18}, respectively. The observed shift of the curves in the vertical direction corresponds to the factor α.

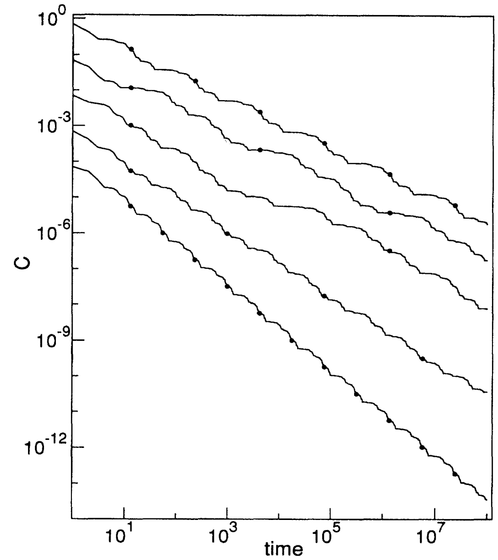

FIG. 3. Amplitude C vs time for the critical parameter value $\lambda = 1$ and different initial phases: from top to bottom $\theta^0 = 1/4$; $(3\sqrt{5}-5)/10$; $1/3$; $(5\sqrt{5}-7)/19$; $(\sqrt{5}-1)/2$. (For all calculations the initial conditions were $x = 1$, we have shifted the curves for clarity.) The appearing periodic structures of C, denoted by dots, correspond to the periods in Table I.

So far we have considered the system exactly at the critical point. The scaling of λ near λ_c is evident from the multiplicative nature of this parameter. Near the threshold we can write $\lambda = \exp(\varepsilon)$ with $|\varepsilon| \ll 1$, and after T iterations the effective parameter value is $\lambda_{\text{eff}} = \lambda^T = \exp(T\varepsilon)$. Thus ε is renormalized, like inverse time, with the factor of W^N.

As a next step we check the obtained scaling properties at the onset of the strange nonchaotic behavior by direct numerical calculations. To illustrate the quantitative self-similarity near the initial phase $\theta^0 = 1/4$ we plotted the phase portraits of the attractor (Fig. 1). It is demonstrated that we get similar pictures by an appropriate rescaling for θ and $\lambda - \lambda_c$ with a factor W^6, and x with a factor α, respectively. For the same parameter values the temporal behavior of the amplitude C is illustrated in Fig. 2. Using the logarithmic scale for the graphs of C vs rescaled time it becomes obvious that the plots reproduce each other with a constant shift corresponding to the factor α. In Fig. 3 we show the dynamics of the amplitude C at the critical point $\lambda = 1$ for different initial phases. As expected, the value of C decreases according to a power law. The values of the slopes computed from the data of Fig. 3 are in good agreement with the theoretical values of γ summarized in Table I. Note that the periodicity of the curves in Fig. 3 reproduces precisely the periods of the renormalization transformation. To emphasize this we have marked the periods by dots.

We have also reproduced the same calculations for the map (1) with the nonlinear function $\tanh(x)$. The same scaling properties are observed in this case, so we conjecture that both maps belong to the same universality class.

Note that the dependence of the scaling constant α on the initial phase θ^0 means that in fact the strange nonchaotic attractor is a multifractal, with different local scalings at different points. The multifractal properties of a strange nonchaotic attractor will be discussed elsewhere [17].

In conclusion, we have developed a renormalization group approach that describes the scaling properties of a strange nonchaotic attractor near the point of its appearance. The models considered here have rather specific symmetry properties. Recent studies show, however, that other mechanisms may be responsible for transitions to strange nonchaotic attractors [6]. It remains an outstanding question whether the approach presented above can be extended to other cases. A similar renormalization procedure has recently been applied to a quasiperiodically forced quantum system [19].

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